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Dedicated to S. V. Shugrin

VARIATIONAL APPROACH TO CONSTRUCTING HYPERBOLIC MODELS OF TWO-VELOCITY MEDIA

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A generalized Hamilton variational principle of the mechanics of two-velocity media is proposed, and equations of motion for homogeneous and heterogeneous two-velocity continua are formulated. It is proved that the convexity of internal energy ensures the hyperbolicity of the one-dimensional equations of motion of such media linearized for the state of rest. In this case, the internal energy is a function of both the phase densities and the modulus of the difference in velocity between the phases. For heterogeneous media with incompressible components, it is shown that, in the case of low volumetric concentrations, the dependence of the internal energy on the modulus of relative velocity ensures the hyperbolicity of the equations of motion for any relative velocity of motion of the phases.

Introduction. At least three approaches to constructing mathematical models of two-velocity media are known at present. The averaging method is used most widely, especially to constructing models of motion for heterogeneous two-velocity media. A distinguishing feature of heterogeneous media is that each phase occupies only part of the volume of the mixture, unlike in homogeneous mixtures, in which each phase is uniformly distributed over the entire volume of the mixture. Applying an appropriate averaging operator to the equations of conservation of mass, momentum, etc. that are valid within each phase, one obtains averaged equations of motion. The main problem that arises in this approach consists in closing the resulting system: the system contains more unknowns than equations. Different experimental and theoretical assumptions on the flow structure, the mechanism of interaction between the phases, etc. [1–3] (see also a review [4]), are used for closing. As was noted by many authors [5, 6], if the pressure in the phases coincide, the corresponding equations of motion in a nondissipative approximation turn out to be *nonhyperbolic* even when the relative difference between the phase velocities is slight. This means that the Cauchy problem for the corresponding nonlinear equations of motion is incorrect.

In [7–9], hyperbolic (nonequilibrium in pressure) models of two-layer liquid flows were obtained by the averaging method. For closure of the equations of motion, a series of hypotheses relating the pressure and velocity at the interface between the liquids to their average values in the layer were invoked [7], or the process of mixing of the liquid at the interface was taken into account by introducing a third liquid layer [9]. An interesting two-velocity model of a bubble liquid which takes into account oscillations of bubbles is proposed in [10]. In the approximation of an incompressible liquid and a small bubble concentration, the model is hyperbolic for a low relative velocity of the bubbles and gives steady wave modes. Interesting hyperbolic models are also proposed in [11–14]. Their hyperbolicity was reached using closing relations, which are usually

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specific to the type of flow or by introducing additional "artificial" terms in equations to predict the correct mechanism of interphase interaction.

The second method, known as the Landau *method of conservation laws*, was originally used to construct models of quantum liquids — superfluid helium [15–17]. The essence of this method is that the requirement that the laws of conservation of mass, momentum, energy, and entropy be satisfied, supplemented by the Galilean principle of relativity, allow one to completely determine the equations of motion of a two-velocity medium. Recently, this approach was used to construct the equations of motion of classical liquids — two-velocity hydrodynamics [18–22]. In particular, in [19], the corresponding equations of motion of a two-liquid medium in a nondissipative approximation are also hyperbolic for the case where the pressures in the phases coincide.

Finally, an effective method of obtaining the equations of motion of two-velocity media is the *variational method* [23–29]. As a rule, the varied functional is the operation after Hamilton: the Lagrangian of a system is the difference between the kinetic and potential energies of the system. Actually, it is not possible to separate the total energy of a two-velocity continuum into kinetic and potential energies. Formal separation of the energy into kinetic and internal energies is ambiguous but, at any definition, the internal energy is a Galilean scalar, which can depend not only on thermodynamic variables but also on the modulus of the velocity of relative motion of the phases $\mathbf{w}$. In this case, the variational approach changes significantly. The Lagrangian of the system must be formulated as the difference between the kinetic energy and the thermodynamic potential, which is related to the internal energy by a partial Legendre transform with respect to the variable $\mathbf{w}$.

The fact that the Lagrangian of a two-velocity system must include additional terms that depend on the modulus of the relative velocity was pointed out in some particular cases (flow of a liquid with gas bubbles) in [24, 26–28]. The hyperbolicity of the resulting equations for a low relative velocity of phases is proved in [27, 28].

We derive the equation of motion for a two-velocity continuum on the basis of the generalized Hamilton principle, requiring convexity of the internal energy with respect to the sought variables [30]. In particular, convexity of internal energy, indicating thermodynamic stability of the medium, ensures hyperbolicity of the equations of motion of a two-velocity medium linearized on a zero hydrodynamic background.

1. Variational Approach to Describing Two-Velocity Media. We consider a two-liquid medium that is characterized by velocities $\mathbf{u}_1$ and $\mathbf{u}_2$ and densities ρ_1 and ρ_2 of its components and internal energy U. A fundamental difference between a two-velocity medium and a single-velocity medium is the dependence of internal energy on velocity: the internal energy U depends on the modulus of the Galilean invariant — the relative velocity $\mathbf{w} = \mathbf{u}_2 - \mathbf{u}_1$. The presence of this dependence changes the variational principle qualitatively.

Elementary Example. We consider a weight on a nonlinear spring. The equations of motion have the form

$$\frac{d}{dt}p(\dot{x}) + \frac{\partial F(x)}{\partial x} = 0,$$

where $p(\dot{x})$ is the momentum of the weight and $F(x)$ is the potential of the spring. The integral of the energy and the Lagrangian have the form

$$E = \dot{x}p(\dot{x}) - \int^{\dot{x}} p(y)\,dy + F(x), \qquad L(x,\dot{x}) = \int^{\dot{x}} p(y)\,dy - F(x).$$

It can easily be seen that they are related to one another by a partial Legendre transform with respect the variable $\dot{x}$:

$$E = \dot{x}L_{\dot{x}} - L.$$

From the last relation it follows that the natural variables for the energy E are the variables x and p, and not the variables x and $\dot{x}$, $E = E(x,p)$.

Variational Principle of the Mechanics of a Two-Velocity Medium. The total energy of a two-velocity system is written in standard form $E = \rho_1|\mathbf{u}_1|^2/2 + \rho_2|\mathbf{u}_2|^2/2 + U$.

The energy of the system E is usually divided into kinetic and internal energies. This is achieved by conversion to a moving system in which the elementary volume of the continuum is at rest. The total energy of the medium in this system is assumed to be the internal energy of the medium U. For two-velocity media there is no coordinate system in which motion can be eliminated. As a consequence, the use of the standard definition of internal energy leads to the dependence of the Galilean invariant — the modulus of the difference between the velocities of the medium component $\mathbf{w} = \mathbf{u}_2 - \mathbf{u}_1$.

Thus, the internal energy U of a two-velocity medium is a function of additive independent thermodynamic parameters, including the relative momentum $\mathbf{i}$ — a Galilean invariant thermodynamic variable that is conjugate to the velocity $\mathbf{w}$:

$$U = U(\rho_1, \rho_2, |\mathbf{i}|).$$

In the adiabatic approximation adopted there is no dependence on entropy.

Using a Legendre transform, it is possible to introduce the thermodynamic potential $W = W(\rho_1, \rho_2, |\mathbf{w}|)$:

$$W = U - (\mathbf{i}, \mathbf{w}), \qquad \mathbf{w} = \frac{\partial U}{\partial \mathbf{i}}. \tag{1.1}$$

Knowing the thermodynamic potential W as a function of ρ_1, ρ_2, and $|\mathbf{w}|$, it is possible to define the internal energy of the system:

$$U = W + (\mathbf{i}, \mathbf{w}) = W - \left(\frac{\partial W}{\partial \mathbf{w}}, \mathbf{w}\right), \qquad \mathbf{i} = -\frac{\partial W}{\partial \mathbf{w}}. \tag{1.2}$$

Definitions (1.1) and (1.2) can be represented as

$$W = U - iw, \quad U = W + iw, \quad i = |\mathbf{i}|, \quad w = |\mathbf{w}|, \quad i = -\frac{\partial W}{\partial w}, \quad w = \frac{\partial U}{\partial i}.$$

The example of the weight–spring system shows that in order to formulate an analog of the Hamilton principle of least action for a two-velocity continuum, it is necessary to consider the Lagrangian of the two-velocity system

$$\tilde{L} = \rho_1|\mathbf{u}_1|^2/2 + \rho_2|\mathbf{u}_2|^2/2 - W(\rho_1, \rho_2, |\mathbf{w}|). \tag{1.3}$$

Lagrangian (1.3) allows one to formulate the variational principle of the mechanics of a two-velocity medium:

$$\delta \int_{t_1}^{t_2}\int_{R^n} \left(\frac{\rho_1|\mathbf{u}_1|^2}{2} + \frac{\rho_2|\mathbf{u}_2|^2}{2} - W(\rho_1, \rho_2, |\mathbf{w}|)\right) d\mathbf{x}\, dt = 0. \tag{1.4}$$

Generalized Variational Principle. In the general case, the internal energy of complex media U can depend on the time derivatives of thermodynamic variables, the modulus of the Galilean invariant $\mathbf{w}$, etc. (see, for example, [31], where examples of media whose internal energy depends on the total derivative of density with respect are given.

The variational principle for these media should be formulated on the basis of a functional that represents the difference between the kinetic energy of the system K and the thermodynamic potential W. The latter is related to the internal energy of the system U by a partial Legendre transform with respect to the variables ζ, which are conjugate to the time derivatives:

$$\delta \int_{t_1}^{t_2}\int_{R^n} (K - W) d\mathbf{x}\, dt = 0, \qquad U = W - \frac{\partial W}{\partial \zeta}\zeta.$$

2. Variational Approach to Describing Two-Velocity Homogeneous Media. We consider two mixed liquids whose motion is characterized by their velocities $\mathbf{u}_1$ and $\mathbf{u}_2$, densities ρ_1 and ρ_2, and internal energy U. The word "mixed" means that the volumetric concentration of the material is not a sought parameter, i.e., each component occupies the entire volume of the mixture equivalently with the others. Such multiphase media are called *homogeneous* media.

We consider the case of a mechanical system where the entropies of the media are not sought parameters. In addition, we ignore the dependence of energy on derivative thermodynamic values. We are interested in the equations of motion of the medium in which the internal energy depends on the relative velocity.

Variational Principle and Conservation Laws. The Hamilton principle of least action for a two-velocity homogeneous continuum is written in the form (1.4):

$$\delta \int_{t_1}^{t_2}\int_{R^n} \left(\frac{1}{2}\rho_1|\mathbf{u}_1|^2 + \frac{1}{2}\rho_2|\mathbf{u}_2|^2 - W(\rho_1,\rho_2,w)\right) d\mathbf{x}\, dt = 0. \tag{2.1}$$

Here the potential $W(\rho_1,\rho_2,w)$ is related to the volumetric internal energy of the system $U(\rho_1,\rho_2,i)$ by the partial Legendre transform (1.1) and (1.2).

As restrictions to (2.1) we add the equations of conservation of mass for each phase:

$$\mathcal{M}_1 = \frac{\partial \rho_1}{\partial t} + \operatorname{div}(\rho_1\mathbf{u}_1) = 0, \qquad \mathcal{M}_2 = \frac{\partial \rho_2}{\partial t} + \operatorname{div}(\rho_2\mathbf{u}_2) = 0. \tag{2.2}$$

Introducing Lagrangian multipliers $\varphi_1(t,\mathbf{x})$ and $\varphi_2(t,\mathbf{x})$, we consider the Lagrangian of our system L, determined with accuracy up to the divergent term:

$$L = \rho_1\left(\frac{1}{2}|\mathbf{u}_1|^2 - \frac{d_1\varphi_1}{dt}\right) + \rho_2\left(\frac{1}{2}|\mathbf{u}_2|^2 - \frac{d_2\varphi_2}{dt}\right) - W(\rho_1,\rho_2,w). \tag{2.3}$$

Here the operators d_i/dt are given by the formulas

$$\frac{d_i}{dt} = \frac{\partial}{\partial t} + (\mathbf{u}_i,\nabla). \tag{2.4}$$

For simplicity, we shall use Cartesian coordinates $\{x^k\}$, $k = 1, 2, 3$. Summation is performed over recurring indices.

For each phase, we introduce Lagrangian coordinates $\{\xi^a\}$ and $\{\eta^b\}$ (a, $b = 1, 2, 3$):

$$\frac{\partial \xi^a}{\partial t} + u_1^k\frac{\partial \xi^a}{\partial x^k} = 0, \qquad \frac{\partial \eta^b}{\partial t} + u_2^k\frac{\partial \eta^b}{\partial x^k} = 0. \tag{2.5}$$

We designate

$$\xi_t^a = \frac{\partial \xi^a}{\partial t}, \quad \eta_t^b = \frac{\partial \eta^b}{\partial t}, \quad \xi_{,k}^a = \frac{\partial \xi^a}{\partial x^k}, \quad \eta_{,k}^b = \frac{\partial \eta^b}{\partial x^k}, \quad x_{,a(\xi)}^k = \frac{\partial x^k}{\partial \xi^a}, \quad x_{,b(\eta)}^k = \frac{\partial x^k}{\partial \eta^a}. \tag{2.6}$$

Where the context is clear, we shall write $x_{,a}^k$ or $x_{,b}^k$, omitting the dependence on ξ or η. From (2.5), (2.6) we obtain the relations

$$\begin{aligned} u_1^k &= -\xi_t^a x_{,a(\xi)}^k, \qquad \frac{\partial u_1^k}{\partial \xi_t^a} = -x_{,a(\xi)}^k, \qquad \frac{\partial u_1^k}{\partial \xi_{,j}^a} = -u_1^j x_{,a(\xi)}^k, \\ u_2^k &= -\eta_t^b x_{,b(\eta)}^k, \qquad \frac{\partial u_2^k}{\partial \eta_t^b} = -x_{,b(\eta)}^k, \qquad \frac{\partial u_2^k}{\partial \eta_{,j}^a} = -u_2^j x_{,b(\eta)}^k. \end{aligned} \tag{2.7}$$

We consider the Lagrangian L given by formula (2.3) as a function of the variables ξ_t^a, $\xi_{,k}^a$, η_t^b, $\eta_{,k}^b$, φ_{1t}, $\varphi_{1,k}$, φ_{2t}, $\varphi_{2,k}$, ρ_1, and ρ_2. The subscript t denotes the partial time derivative $\partial/\partial t$.

We define the variational derivatives

$$L_a \equiv \frac{\delta L}{\delta \xi^a} = -\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \xi_t^a}\right) - \frac{\partial}{\partial x^k}\left(\frac{\partial L}{\partial \xi_{,k}^a}\right), \qquad L_b \equiv \frac{\delta L}{\delta \eta^b} = -\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \eta_t^b}\right) - \frac{\partial}{\partial x^k}\left(\frac{\partial L}{\partial \eta_{,k}^b}\right),$$

$$L_{\varphi_i} \equiv \frac{\delta L}{\delta \varphi_i} = \frac{\partial \rho_i}{\partial t} + \frac{\partial}{\partial x^k}(\rho_i u_i^k), \qquad L_{\rho_i} \equiv \frac{\delta L}{\delta \rho_i} = \frac{1}{2}|\mathbf{u}_i|^2 - \frac{d_i\varphi_i}{dt} - \frac{\partial W}{\partial \rho_i}.$$

Equating them to zero and eliminating Lagrangian multipliers, we obtain the required equations of motion.

The invariance of the Lagrangian with respect to shifts in spatial variables leads, according to the Noether theorem [32], to divergence of the expression

$$\partial_m \equiv \xi^a_{,m} L_a + \eta^b_{,m} L_b + \varphi_{1,m} L_{\varphi_1} + \varphi_{2,m} L_{\varphi_2} + \rho_{1,m} L_{\rho_1} + \rho_{2,m} L_{\rho_2}$$

$$= \frac{\partial}{\partial t}\Big(- \xi^a_{,m} \frac{\partial L}{\partial \xi^a_t} - \eta^b_{,m} \frac{\partial L}{\partial \eta^b_t} - \varphi_{1,m} \frac{\partial L}{\partial \varphi_{1t}} - \varphi_{2,m} \frac{\partial L}{\partial \varphi_{2t}}\Big)$$

$$+ \frac{\partial}{\partial x^k}\Big(- \xi^a_{,m} \frac{\partial L}{\partial \xi^a_{,k}} - \eta^b_{,m} \frac{\partial L}{\partial \eta^b_{,k}} - \varphi_{1,m} \frac{\partial L}{\partial \varphi_{1,k}} - \varphi_{2,m} \frac{\partial L}{\partial \varphi_{2,k}} + L\delta^k_m\Big). \tag{2.8}$$

The invariance of the Lagrangian with respect to shifts in time leads to divergence of the expression (the minus sign is used for convenience):

$$\mathcal{E} \equiv - \xi^a_t L_a - \eta^b_t L_b - \varphi_{1t} L_{\varphi_1} - \varphi_{2t} L_{\varphi_2} - \rho_{1t} L_{\rho_1} - \rho_{2t} L_{\rho_2}$$

$$= \frac{\partial}{\partial t}\Big(\xi^a_t \frac{\partial L}{\partial \xi^a_t} + \eta^b_t \frac{\partial L}{\partial \eta^b_t} + \varphi_{1t} \frac{\partial L}{\partial \varphi_{1t}} + \varphi_{2t} \frac{\partial L}{\partial \varphi_{2t}} - L\Big)$$

$$+ \frac{\partial}{\partial x^k}\Big(\xi^a_t \frac{\partial L}{\partial \xi^a_{,k}} + \eta^b_t \frac{\partial L}{\partial \eta^b_{,k}} + \varphi_{1t} \frac{\partial L}{\partial \varphi_{1,k}} + \varphi_{2t} \frac{\partial L}{\partial \varphi_{2,k}}\Big). \tag{2.9}$$

The divergence of (2.8), (2.9) can be verified by direct calculations.

Since the variational derivatives vanish, Eqs. (2.8) and (2.9) are conservation laws: Eq. (2.8) is the law of conservation of the momentum $\mathbf{j} = \rho_1 \mathbf{u}_1 + \rho_2 \mathbf{u}_2$ and Eq. (2.9) is the law of conservation of the energy E. We derive an explicit form of Eqs. (2.8) and (2.9) in terms of the sought variables.

From the definition of the variational derivatives, we obtain

$$\rho_1 L_{\rho_1} + \rho_2 L_{\rho_2} = \rho_1 \Big(\frac{1}{2}|\mathbf{u}_1|^2 - \frac{d_1\varphi_1}{dt}\Big) + \rho_2 \Big(\frac{1}{2}|\mathbf{u}_2|^2 - \frac{d_2\varphi_2}{dt}\Big)$$

$$- \rho_1 \frac{\partial W}{\partial \rho_1} - \rho_2 \frac{\partial W}{\partial \rho_2} + W - W = L - \Big(\rho_1 \frac{\partial W}{\partial \rho_1} + \rho_2 \frac{\partial W}{\partial \rho_2} - W\Big).$$

Hence it follows that on the extremal the following equality holds:

$$L = \rho_1 \frac{\partial W}{\partial \rho_1} + \rho_2 \frac{\partial W}{\partial \rho_2} - W. \tag{2.10}$$

We designate $w^k = u^k_2 - u^k_1$. Using relation (2.7), we have

$$-\xi^a_{,m} \frac{\partial L}{\partial \xi^a_t} = -\xi^a_{,m} \Big\{\rho_1 u_{1k}(-x^k_{,a}) - \frac{\partial W}{\partial w^k} x^k_{,a} + \rho_1 \varphi_{1,k} x^k_{,a}\Big\} = \rho_1 u_{1m} + \frac{\partial W}{\partial w^m} - \rho_1 \varphi_{1,m},$$

$$-\eta^b_{,m} \frac{\partial L}{\partial \eta^b_t} = \rho_2 u_{2m} - \frac{\partial W}{\partial w^m} - \rho_2 \varphi_{2,m}, \quad -\varphi_{1,m} \frac{\partial L}{\partial \varphi_{1t}} = \varphi_{1,m}\rho_1, \quad -\varphi_{2,m} \frac{\partial L}{\partial \varphi_{2t}} = \varphi_{2,m}\rho_2,$$

$$-\xi^a_{,m} \frac{\partial L}{\partial \xi^a_{,k}} = -\xi^a_{,m} \Big\{\rho_1 u_{1i}(-u^k_1 x^i_{,a}) - \frac{\partial W}{\partial w^i}(u^k_1 x^i_{,a}) - \rho_1 \varphi_{1,i}(-u^k_1 x^i_{,a})\Big\} = \rho_1 u_{1m} u^k_1 + \frac{\partial W}{\partial w^m} u^k_1 - \rho_1 \varphi_{1,m} u^k_1,$$

$$-\eta^b_{,m} \frac{\partial L}{\partial \eta^b_k} = \rho_2 u_{2m} u^k_2 - \frac{\partial W}{\partial w^m} u^k_2 - \rho_2 \varphi_{2,m} u^k_2,$$

$$-\varphi_{1,m} \frac{\partial L}{\partial \varphi_{1,k}} = \rho_1 \varphi_{1,m} u^k_1, \qquad -\varphi_{2,m} \frac{\partial L}{\partial \varphi_{2,k}} = \rho_2 \varphi_{2,m} u^k_2.$$

Then, Eq. (2.8) leads to the following equation for the total momentum of the medium:

$$\partial_m = \frac{\partial}{\partial t}(\rho_1 u_{1m} + \rho_2 u_{2m}) + \frac{\partial}{\partial x^k}\Big(\rho_1 u_{1m} u^k_1 + \rho_2 u_{2m} u^k_2 - \frac{\partial W}{\partial w^m} w^k + L\delta^k_m\Big) = 0.$$

It is obvious that the introduced Lagrangian function L coincides with the thermodynamic definition of the pressure in the system.

Using relation (2.10), we write

$$\vec{\mathfrak{J}} = \frac{\partial}{\partial t}(\rho_1 \mathbf{u}_1 + \rho_2 \mathbf{u}_2) + \operatorname{div}\Big(\rho_1 \mathbf{u}_1 \otimes \mathbf{u}_1 + \rho_2 \mathbf{u}_2 \otimes \mathbf{u}_2 - \frac{\partial W}{\partial \mathbf{w}} \otimes \mathbf{w} + \Big(\rho_1 \frac{\partial W}{\partial \rho_1} + \rho_2 \frac{\partial W}{\partial \rho_2} - W\Big) I\Big) = 0, \quad (2.11)$$

where $\otimes$ is a tensor product and I is a unit tensor.

Similarly,

$$\xi^a_t \frac{\partial L}{\partial \xi^a_t} = \rho_1 |\mathbf{u}_1|^2 + \frac{\partial W}{\partial w^k} u_1^k - \rho_1 \varphi_{1,k} u_1^k, \qquad \varphi_{1t} \frac{\partial L}{\partial \varphi_{1t}} = -\rho_1 \varphi_{1t},$$

$$\eta^b_t \frac{\partial L}{\partial \eta^b_t} = \rho_2 |\mathbf{u}_2|^2 - \frac{\partial W}{\partial w^k} u_2^k - \rho_2 \varphi_{2,k} u_2^k, \qquad \varphi_{2t} \frac{\partial L}{\partial \varphi_{2t}} = -\rho_2 \varphi_{2t},$$

$$\xi^a_t \frac{\partial L}{\partial \xi^a_{,k}} = \rho_1 |\mathbf{u}_1|^2 u_1^k + \frac{\partial W}{\partial w^i} u_1^i u_1^k - \rho_1 \varphi_{1,i} u_1^i u_1^k, \qquad \varphi_{1t} \frac{\partial L}{\partial \varphi_{1,k}} = -\varphi_{1t} \rho_1 u_1^k,$$

$$\eta^b_t \frac{\partial L}{\partial \eta^b_{,k}} = \rho_2 |\mathbf{u}_2|^2 u_2^k - \frac{\partial W}{\partial w^i} u_2^i u_2^k - \rho_2 \varphi_{2,i} u_2^i u_2^k, \qquad \varphi_{2t} \frac{\partial L}{\partial \varphi_{2,k}} = -\varphi_{2t} \rho_2 u_2^k.$$

Substituting the formulas obtained above into (2.9) and using the definition of L, we finally have

$$\mathcal{E} = \frac{\partial}{\partial t}\left(\frac{1}{2}\rho_1 |\mathbf{u}_1|^2 + \frac{1}{2}\rho_2 |\mathbf{u}_2|^2 + W - \mathbf{w}\frac{\partial W}{\partial \mathbf{w}}\right) + \operatorname{div}\Big(\rho_1 \mathbf{u}_1 \Big(\frac{|\mathbf{u}_1|^2}{2} + \frac{\partial W}{\partial \rho_1}\Big)$$

$$+ \rho_2 \mathbf{u}_2 \Big(\frac{|\mathbf{u}_2|^2}{2} + \frac{\partial W}{\partial \rho_2}\Big) - (\mathbf{u}_2 \otimes \mathbf{u}_2 - \mathbf{u}_1 \otimes \mathbf{u}_1)\Big\langle \frac{\partial W}{\partial \mathbf{w}} \Big\rangle\Big) = 0. \quad (2.12)$$

Here the operation $A\langle \mathbf{b} \rangle$ denotes multiplication of the tensor A by the vector $\mathbf{b}$.

The equations of motion of the phases are obtained by a standard method. We consider the expression

$$\xi^a_{,m} L_a \equiv -\xi^a_{,m}\Big(\frac{\partial}{\partial t}\Big(\frac{\partial L}{\partial \xi^a_t}\Big) + \frac{\partial}{\partial x^k}\Big(\frac{\partial L}{\partial \xi^a_{,k}}\Big)\Big) = 0.$$

Direct calculations lead to the equation

$$\frac{\partial}{\partial t}\Big(\rho_1 u_{1m} + \frac{\partial W}{\partial w^m}\Big) + \frac{\partial}{\partial x^k}\Big(\Big(\rho_1 u_{1m} + \frac{\partial W}{\partial w^m}\Big) u_1^k\Big) + \Big(\rho_1 u_{1k} + \frac{\partial W}{\partial w^k}\Big)\frac{\partial u_1^k}{\partial x^m} - \rho_1 \frac{\partial}{\partial x^m}\Big(\frac{d_1 \varphi_1}{dt}\Big) = 0.$$

Since from the relation $L_{\rho_1} = 0$ it follows that

$$\frac{d_1 \varphi_1}{dt} = \frac{1}{2}|\mathbf{u}_1|^2 - \frac{\partial W}{\partial \rho_1},$$

we obtain the following equation for $\mathbf{u}_1$:

$$\rho_1 \frac{\partial u_{1m}}{\partial t} + \rho_1 u_1^k \frac{\partial u_{1m}}{\partial x^k} + \frac{\partial}{\partial t}\Big(\frac{\partial W}{\partial w^m}\Big) + \frac{\partial}{\partial x^k}\Big(\frac{\partial W}{\partial w^m} u_1^k\Big) + \frac{\partial W}{\partial w^k}\frac{\partial u_1^k}{\partial x^m} + \rho_1 \frac{\partial}{\partial x^m}\Big(\frac{\partial W}{\partial \rho_1}\Big) = 0.$$

This equation in vector form is written as

$$\rho_1\Big(\frac{\partial \mathbf{u}_1}{\partial t} + (\mathbf{u}_1 \cdot \nabla)\mathbf{u}_1\Big) - \frac{\partial \mathbf{i}}{\partial t} - \operatorname{div}(\mathbf{i} \otimes \mathbf{u}_1) - \nabla \mathbf{u}_1 \langle \mathbf{i} \rangle + \rho_1 \nabla\Big(\frac{\partial W}{\partial \rho_1}\Big) = 0. \quad (2.13)$$

Here

$$\mathbf{i} = -\frac{\partial W}{\partial \mathbf{w}}, \qquad \nabla \mathbf{u}_1 = \Big(\frac{\partial \mathbf{u}_1}{\partial \mathbf{x}}\Big)^*$$

(the superscript asterisk denotes conjugate mapping).

Similarly, for the second phase, we obtain

$$\rho_2 \frac{\partial u_{2m}}{\partial t} + \rho_2 u_2^k \frac{\partial u_{2m}}{\partial x^k} - \frac{\partial}{\partial t}\Big(\frac{\partial W}{\partial w^m}\Big) - \frac{\partial}{\partial x^k}\Big(\frac{\partial W}{\partial w^m} u_2^k\Big) - \frac{\partial W}{\partial w^k}\frac{\partial u_2^k}{\partial x^m} + \rho_2 \frac{\partial}{\partial x^m}\Big(\frac{\partial W}{\partial \rho_2}\Big) = 0,$$

or in vector form

$$\rho_2\Big(\frac{\partial \mathrm{u}_2}{\partial t}+(\mathrm{u}_2\cdot\nabla)\mathrm{u}_2\Big)+\frac{\partial \mathrm{i}}{\partial t}+\operatorname{div}(\mathrm{i}\otimes\mathrm{u}_2)+\nabla\mathrm{u}_2\langle\mathrm{i}\rangle+\rho_2\nabla\Big(\frac{\partial W}{\partial\rho_2}\Big)=0. \tag{2.14}$$

We finally write the conservation laws (2.2), (2.11), and (2.12) as the system

$$\frac{\partial\rho_1}{\partial t}+\operatorname{div}(\rho_1\mathrm{u}_1)=0,\qquad \frac{\partial\rho_2}{\partial t}+\operatorname{div}(\rho_2\mathrm{u}_2)=0,$$

$$\frac{\partial}{\partial t}(\rho_1\mathrm{u}_1+\rho_2\mathrm{u}_2)+\operatorname{div}\Big(\rho_1\mathrm{u}_1\otimes\mathrm{u}_1+\rho_2\mathrm{u}_2\otimes\mathrm{u}_2-\frac{\partial W}{\partial\mathrm{w}}\otimes\mathrm{w}+I\Big(\rho_1\frac{\partial W}{\partial\rho_1}+\rho_2\frac{\partial W}{\partial\rho_2}-W\Big)\Big)=0,$$

$$\frac{\partial}{\partial t}\Big(\frac{1}{2}\rho_1|\mathrm{u}_1|^2+\frac{1}{2}\rho_2|\mathrm{u}_2|^2+W-\mathrm{w}\frac{\partial W}{\partial\mathrm{w}}\Big)+\operatorname{div}\Big(\rho_1\mathrm{u}_1\Big(\frac{|\mathrm{u}_1|^2}{2}+\frac{\partial W}{\partial\rho_1}\Big)$$

$$+\rho_2\mathrm{u}_2\Big(\frac{|\mathrm{u}_2|^2}{2}+\frac{\partial W}{\partial\rho_2}\Big)-(\mathrm{u}_2\otimes\mathrm{u}_2-\mathrm{u}_1\otimes\mathrm{u}_1)\Big\langle\frac{\partial W}{\partial\mathrm{w}}\Big\rangle\Big)=0.$$

From (2.12), in particular, it follows that the definition of the variational principle in the form (1.4) gives a correct definition of the total energy of the system:

$$E=\frac{1}{2}\rho_1|\mathrm{u}_1|^2+\frac{1}{2}\rho_2|\mathrm{u}_2|^2+U(\rho_1,\rho_2,i),\qquad U(\rho_1,\rho_2,i)=W-wi,\quad i=-\frac{\partial W}{\partial w}.$$

Let $U(\rho_1,\rho_2,i)$ satisfy the condition of thermodynamic stability of the medium.

Condition S: *The function $U(\rho_1,\rho_2,i)$ is a convex function of its variables.*

As a consequence, $W(\rho_1,\rho_2,w)$ is convex with respect to the variables ρ_1 and ρ_2 and concave with respect to the variable w.

System of Equations for Plane Waves. In the one-dimensional case, the system of conservation laws (2.2), (2.11), and (2.12) is closed if the internal energy of the medium is specified. We investigate the type of this system. We designate $u_1^1=u_1$, $u_2^1=u_2$, $x^1=x$, and $w\doteq u_2-u_1$; the subscript t corresponds to $\partial/\partial t$ and the subscript x corresponds to $\partial/\partial x$.

We adopt a simplifying assumption on the form of the internal energy $U(\rho_1,\rho_2,i)$.

Condition A: *The function $U(\rho_1,\rho_2,i)$ has the form*

$$U(\rho_1,\rho_2,i)=\varepsilon(\rho_1,\rho_2)+\frac{i^2}{2a}=\varepsilon(\rho_1,\rho_2)+\frac{aw^2}{2}$$

(*a is a positive constant*).

From condition A it follows that

$$W=\varepsilon(\rho_1,\rho_2)-\frac{aw^2}{2}.$$

Then, from (2.13) and (2.14) we have

$$\rho_1\frac{d_1u_1}{dt}-a\frac{d_1w}{dt}-2awu_{1x}+\rho_1(\varepsilon_1)_x=0; \tag{2.15}$$

$$\rho_2\frac{d_2u_2}{dt}+a\frac{d_2w}{dt}+2awu_{2x}+\rho_2(\varepsilon_2)_x=0, \tag{2.16}$$

where $\varepsilon_i=\partial\varepsilon/\partial\rho_i$.

Thus, with allowance for (2.15) and (2.16), the desired equations for one-dimensional motions with plane waves take the form

$$\begin{gathered}\rho_{1t}+u_1\rho_{1x}+u_{1x}\rho_1=0,\qquad \rho_{2t}+u_2\rho_{2x}+u_{2x}\rho_2=0,\\ \rho_1(u_{1t}+u_1u_{1x})-a(w_t+u_1w_x)-2awu_{1x}+\rho_1(\varepsilon_{11}\rho_{1x}+\varepsilon_{12}\rho_{2x})=0,\\ \rho_2(u_{2t}+u_2u_{2x})+a(w_t+u_2w_x)+2awu_{2x}+\rho_2(\varepsilon_{12}\rho_{1x}+\varepsilon_{22}\rho_{2x})=0,\end{gathered} \tag{2.17}$$

where $\varepsilon_{ij}=\partial^2\varepsilon/\partial\rho_i\partial\rho_j$.

Hyperbolicity of the Equations of Motion of Two-Velocity Homogeneous Media. System (2.17) can be written in matrix form

$$A\mathbf{u}_t + B\mathbf{u}_x = 0,$$

where

$$\mathbf{u} = (\rho_1, \rho_2, u_1, u_2)^t, \qquad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \rho_1 + a & -a \\ 0 & 0 & -a & \rho_2 + a \end{pmatrix},$$

$$B = \begin{pmatrix} u_1 & 0 & \rho_1 & 0 \\ 0 & u_2 & 0 & \rho_2 \\ \rho_1\varepsilon_{11} & \rho_1\varepsilon_{12} & (\rho_1 + 3a)u_1 - 2au_2 & -au_1 \\ \rho_2\varepsilon_{12} & \rho_2\varepsilon_{22} & -au_2 & (\rho_2 + 3a)u_2 - 2au_1 \end{pmatrix}.$$

It is easy to compute that

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (\rho_2 + a)/\delta & a/\delta \\ 0 & 0 & a/\delta & (\rho_1 + a)/\delta \end{pmatrix}, \qquad C = A^{-1}B = \begin{pmatrix} u_1 & 0 & \rho_1 & 0 \\ 0 & u_2 & 0 & \rho_2 \\ m_{11} & m_{12} & d_{11} & d_{12} \\ m_{21} & m_{22} & d_{21} & d_{22} \end{pmatrix},$$

where

$$\delta = (\rho_1 + a)(\rho_2 + a) - a^2 > 0,$$

$$M_{11} = \frac{\rho_1(\rho_2 + a)}{\delta}\varepsilon_{11} + \frac{a\rho_2}{\delta}\varepsilon_{12}, \qquad M_{12} = \frac{\rho_1(\rho_2 + a)}{\delta}\varepsilon_{12} + \frac{a\rho_2}{\delta}\varepsilon_{22},$$

$$m_{21} = \frac{a\rho_1}{\delta}\varepsilon_{11} + \frac{\rho_2(\rho_1 + a)}{\delta}\varepsilon_{12}, \qquad m_{22} = \frac{a\rho_1}{\delta}\varepsilon_{12} + \frac{\rho_2(\rho_1 + a)}{\delta}\varepsilon_{22},$$

$$d_{11} = u_2 - \frac{(\rho_2 + a)(\rho_1 + 3a)w}{\delta}, \qquad d_{12} = \frac{a(\rho_2 + 3a)w}{\delta},$$

$$d_{21} = \frac{a(\rho_1 + 3a)w}{\delta}, \qquad d_{22} = u_1 + \frac{(\rho_1 + a)(\rho_2 + 3a)w}{\delta}.$$

The eigenvalues of the matrix C are determined by solving the equation $\det|C - \lambda I| = 0$. We consider a simplified form of the corresponding fourth-order polynomial in the variable λ obtained by linearization of the system in a neighborhood of the point $u_1^0 = u_2^0$, ρ_1^0, ρ_2^0. Without loss of generality, we assume that $u_1^0 = u_2^0 = 0$ (below, the superscript 0 is omitted). Since d_{ij} vanishes in the linearization, the polynomial takes the form

$$\lambda^4 - \lambda^2(\rho_2 m_{22} + \rho_1 m_{11}) + \rho_1\rho_2(m_{11}m_{22} - m_{21}m_{12}) = 0. \tag{2.18}$$

To confirm that all roots of Eq. (2.18) are real, it suffices to verify that

$$\rho_2 m_{22} + \rho_1 m_{11} > 0, \qquad m_{11}m_{22} - m_{21}m_{12} > 0,$$

$$(\rho_2 m_{22} + \rho_1 m_{11})^2 - 4\rho_1\rho_2(m_{11}m_{22} - m_{21}m_{12}) > 0.$$

The first inequality is verified directly:

$$\rho_2 m_{22} + \rho_1 m_{11} = \frac{a\rho_1\rho_2}{\delta}\varepsilon_{12} + \frac{{\rho_2}^2(\rho_1 + a)}{\delta}\varepsilon_{22} + \frac{a\rho_1\rho_2}{\delta}\varepsilon_{12} + \frac{{\rho_1}^2(\rho_2 + a)}{\delta}\varepsilon_{11} = \frac{\rho_1{\rho_2}^2\varepsilon_{22} + \rho_2{\rho_1}^2\varepsilon_{11}}{\delta} + \frac{a}{\delta}(\varepsilon\rho, \rho) > 0,$$

where $(\varepsilon\rho, \rho)$ denotes the positive definite quadratic form

$$(\varepsilon\rho, \rho) = \varepsilon_{11}{\rho_1}^2 + 2\varepsilon_{12}\rho_1\rho_2 + \varepsilon_{22}{\rho_2}^2.$$

The second inequality is also verified by direct calculations:

$$m_{11}m_{22} - m_{12}m_{21} = \Big(\frac{\rho_1(\rho_2+a)}{\delta}\varepsilon_{11} + \frac{a\rho_2}{\delta}\varepsilon_{12}\Big)\Big(\frac{a\rho_1}{\delta}\varepsilon_{12} + \frac{\rho_2(\rho_1+a)}{\delta}\varepsilon_{22}\Big)$$

$$- \Big(\frac{\rho_1(\rho_2+a)}{\delta}\varepsilon_{12} + \frac{a\rho_2}{\delta}\varepsilon_{22}\Big)\Big(\frac{a\rho_1}{\delta}\varepsilon_{11} + \frac{\rho_2(\rho_1+a)}{\delta}\varepsilon_{12}\Big)$$

$$= \Big(\frac{\rho_1\rho_2(\rho_1+a)(\rho_2+a)}{\delta^2} - \frac{a^2\rho_1\rho_2}{\delta^2}\Big)\varepsilon_{11}\varepsilon_{22} + \Big(\frac{a^2\rho_1\rho_2}{\delta^2} - \frac{\rho_1\rho_2(\rho_1+a)(\rho_2+a)}{\delta^2}\Big)\varepsilon_{12}^2 = \frac{\rho_1\rho_2}{\delta}(\varepsilon_{11}\varepsilon_{22} - \varepsilon_{12}^2) > 0.$$

And, finally,

$$(\rho_2 m_{22} + \rho_1 m_{11})^2 - 4\rho_1\rho_2(m_{11}m_{22} - m_{21}m_{12})$$

$$= \Big(\frac{\rho_1{\rho_2}^2\varepsilon_{22} + \rho_2{\rho_1}^2\varepsilon_{11}}{\delta} + \frac{a}{\delta}({\rho_1}^2\varepsilon_{11} + 2\rho_1\rho_2\varepsilon_{12} + {\rho_2}^2\varepsilon_{22})\Big)^2 - \frac{4{\rho_1}^2{\rho_2}^2}{\delta}(\varepsilon_{11}\varepsilon_{22} - \varepsilon_{12}^2)$$

$$= \frac{{\rho_1}^2{\rho_2}^2}{\delta}\Big(\frac{a^2}{{\rho_1}^2{\rho_2}^2}(\varepsilon\rho,\rho)^2 + 2a\Big(\frac{(\varepsilon\rho,\rho)}{\rho_1\rho_2}(\rho_2\varepsilon_{22} + \rho_1\varepsilon_{11}) - 2(\rho_1+\rho_2)(\varepsilon_{11}\varepsilon_{22} - \varepsilon_{12}^2)\Big)$$

$$+ (\rho_2\varepsilon_{22} + \rho_1\varepsilon_{11})^2 - 4\rho_1\rho_2(\varepsilon_{11}\varepsilon_{22} - \varepsilon_{12}^2)\Big) \equiv \frac{{\rho_1}^2{\rho_2}^2}{\delta} f(a),$$

where $f(a)$ denotes the second-order polynomial in a

$$f(a) = \frac{a^2(\varepsilon\rho,\rho)^2}{{\rho_1}^2{\rho_2}^2} + (\rho_2\varepsilon_{22} + \rho_1\varepsilon_{11})^2 - 4\rho_1\rho_2(\varepsilon_{11}\varepsilon_{22} - \varepsilon_{12}^2)$$

$$+ \frac{2a(\varepsilon\rho,\rho)}{\rho_1\rho_2}\Big(\rho_2\varepsilon_{22} + \rho_1\varepsilon_{11} - \frac{2(\rho_1+\rho_2)\rho_1\rho_2(\varepsilon_{11}\varepsilon_{22} - \varepsilon_{12}^2)}{(\varepsilon\rho,\rho)}\Big)$$

$$= \Big(\frac{a(\varepsilon\rho,\rho)}{\rho_1\rho_2} + \rho_2\varepsilon_{22} + \rho_1\varepsilon_{11} - \frac{2(\rho_1+\rho_2)\rho_1\rho_2(\varepsilon_{11}\varepsilon_{22} - \varepsilon_{12}^2)}{(\varepsilon\rho,\rho)}\Big)^2$$

$$+ \frac{4(\rho_1+\rho_2)\rho_1\rho_2(\rho_1\varepsilon_{11} + \rho_2\varepsilon_{22})(\varepsilon_{11}\varepsilon_{22} - \varepsilon_{12}^2)}{(\varepsilon\rho,\rho)}$$

$$- \frac{4(\rho_1+\rho_2)^2{\rho_1}^2{\rho_2}^2(\varepsilon_{11}\varepsilon_{22} - \varepsilon_{12}^2)^2}{(\varepsilon\rho,\rho)^2} - 4\rho_1\rho_2(\varepsilon_{11}\varepsilon_{22} - \varepsilon_{12}^2).$$

If the function $g(\rho_1\rho_2) = 4((\rho_1+\rho_2)(\rho_1\varepsilon_{11}+\rho_2\varepsilon_{22})(\varepsilon\rho,\rho) - (\varepsilon\rho,\rho)^2 - \rho_1\rho_2(\rho_1+\rho_2)^2(\varepsilon_{11}\varepsilon_{22} - \varepsilon_{12}^2))\rho_1\rho_2(\varepsilon_{11}\varepsilon_{22} - \varepsilon_{12}^2)/(\varepsilon\rho,\rho)^2$ is nonnegative for all $\rho_1, \rho_2 > 0$, then $f(a)$ is nonnegative for all $a \geqslant 0$. The latter is proved by the following calculation. We designate $z = \rho_1/\rho_2$. Then, $-(\varepsilon_{11}z^2 + 2\varepsilon_{12}z + \varepsilon_{22})^2 - z(1+z)^2(\varepsilon_{11}\varepsilon_{22} - \varepsilon_{12}^2) + (1+z)(\varepsilon_{11}z + \varepsilon_{22})(\varepsilon_{11}z^2 + 2\varepsilon_{12}z + \varepsilon_{22}) = z(\varepsilon_{12} - \varepsilon_{22} + \varepsilon_{11}z - \varepsilon_{12}z)^2 \geqslant 0$ for $z > 0$. Thus, $g(\rho_1, \rho_2) \geqslant 0$. This implies that our system is hyperbolic for all $a \geqslant 0$.

3. Variational Approach to Describing Two-Velocity Heterogeneous Incompressible Media. Let us derive equations of motion for a heterogeneous two-velocity medium. The components of the medium are assumed to be incompressible. In the adopted adiabatic approximation there is no dependence on entropy.

Variational Principle and Conservation Laws. For the case of incompressible moving media composing a heterogeneous two-velocity medium, there is one more constraint, which corresponds to the condition of compatible deformation of the two components. We introduce the volumetric concentrations of the components $\alpha_1 = \rho_1/\hat{\rho}_1$ and $\alpha_2 = \rho_2/\hat{\rho}_2$ by expressing the partial densities of the two components in terms of the physical densities $\hat{\rho}_1$ and $\hat{\rho}_2$, which are constant by virtue of the incompressibility of the moving media. It is necessary to note that the change in density ρ can be due only to a change in the ratio components 1 and 2 in a unit volume of the two-velocity medium. Then, the additivity of mass leads to the following additional constraint on the volumetric concentrations:

$$\alpha_1 + \alpha_2 = 1. \qquad (3.1)$$

In the formulation of the variational principle, the presence of constraint (3.1) is taken into account like conditions (2.2) and (2.3) using the Lagrange multiplier γ

$$\delta \int_{t_1}^{t_2}\int_{R^n} \left(\frac{\rho_1|\mathbf{u}_1|^2}{2} + \frac{\rho_2|\mathbf{u}_2|^2}{2} - W(\rho_1,\rho_2,|\mathbf{w}|) - \rho_1(\varphi_{1t} + u_1^k\varphi_{1,k}) - \rho_2(\varphi_{2t} + u_2^k\varphi_{2,k}) + \gamma(\alpha_1 + \alpha_2 - 1)\right) d\mathbf{x}\, dt = 0.$$

In what follows, it is more convenient to use the variables ρ_1 and ρ_2 rather than α_1 and α_2. Finally, the Lagrangian takes the form

$$L = \frac{\rho_1|\mathbf{u}_1|^2}{2} + \frac{\rho_2|\mathbf{u}_2|^2}{2} - W(\rho_1,\rho_2,|\mathbf{w}|) - \rho_1(\varphi_{1t} + u_1^k\varphi_{1,k}) - \rho_2(\varphi_{2t} + u_2^k\varphi_{2,k}) + \gamma\Big(\frac{\rho_1}{\hat\rho_1} + \frac{\rho_2}{\hat\rho_2} - 1\Big). \quad (3.2)$$

Converting (3.2) to the Lagrangian variables ξ^a and η^b given by relations (2.5), we find that the Lagrangian L is a function of the variables ρ_1, ρ_2, ξ^a_t, $\xi^a_{,k}$, η^b_t, $\eta^b_{,k}$, φ_{1t}, $\varphi_{1,k}$, φ_{2t}, $\varphi_{2,k}$, and γ.

Variation over the Lagrange multipliers φ_1, φ_2, and γ gives constraints (2.2) and (3.1) imposed on the system:

$$L_{\varphi_1} \equiv \frac{\delta L}{\delta \varphi_1} = \frac{\partial \rho_1}{\partial t} + \operatorname{div}(\rho_1\mathbf{u}_1), \quad L_{\varphi_2} \equiv \frac{\delta L}{\delta \varphi_2} = \frac{\partial \rho_2}{\partial t} + \operatorname{div}(\rho_2\mathbf{u}_2), \quad L_\gamma \equiv \frac{\delta L}{\delta\gamma} = \alpha_1 + \alpha_2 - 1.$$

The variations over the partial densities ρ_1 and ρ_2 have the form

$$L_{\rho_1} \equiv \frac{\delta L}{\delta \rho_1} = \frac{|\mathbf{u}_1|^2}{2} - \frac{\partial W}{\partial \rho_1} - \varphi_{1t} - u_1^k\varphi_{1,k} + \frac{\gamma}{\hat\rho_1}, \qquad L_{\rho_2} \equiv \frac{\delta L}{\delta \rho_2} = \frac{|\mathbf{u}_2|^2}{2} - \frac{\partial W}{\partial \rho_2} - \varphi_{2t} - u_2^k\varphi_{2,k} + \frac{\gamma}{\hat\rho_2}$$

and allow one to determine the total derivatives of the Lagrange multipliers:

$$\frac{d_1\varphi_1}{dt} = \varphi_{1t} + u_1^k\varphi_{1,k} = \frac{|\mathbf{u}_1|^2}{2} - \frac{\partial W}{\partial\rho_1} + \frac{\gamma}{\hat\rho_1}; \quad (3.3)$$

$$\frac{d_2\varphi_2}{dt} = \varphi_{2t} + u_2^k\varphi_{2,k} = \frac{|\mathbf{u}_2|^2}{2} - \frac{\partial W}{\partial\rho_2} + \frac{\gamma}{\hat\rho_2}. \quad (3.4)$$

Using the Lagrange function (3.2), it is possible to introduce a pressure p according to the thermodynamic definition of pressure. Indeed, summing up relations (3.3) and (3.4), multiplied by ρ_1 and ρ_2, respectively, we have

$$p \equiv L = -\gamma - W + \rho_1\frac{\partial W}{\partial \rho_1} + \rho_2\frac{\partial W}{\partial\rho_2}, \quad (3.5)$$

where γ determines the correction to the pressure due to the condition of compatibility of the components.

To obtain the law of conservation of momentum, we deduce an expression similar to (2.8). It characterizes the uniformity of space and has the form of a conservation law:

$$\begin{aligned} \mathcal{Z}_k &= \rho_{1,k}L_{\rho_1} + \rho_{2,k}L_{\rho_2} + \varphi_{1,k}L_{\varphi_1} + \varphi_{2,k}L_{\varphi_2} + \xi^a_{,k}L_a + \eta^b_{,k}L_b + \gamma_{,k}L_\gamma \\ &= \frac{\partial}{\partial t}\Big(-\xi^a_{,k}\frac{\partial L}{\partial \xi^a_t} - \eta^b_{,k}\frac{\partial L}{\partial \eta^b_t} - \varphi_{1,k}\frac{\partial L}{\partial\varphi_{1t}} - \varphi_{2,k}\frac{\partial L}{\partial \varphi_{2t}}\Big) \\ &+ \frac{\partial}{\partial x^m}\Big(-\xi^a_{,k}\frac{\partial L}{\partial \xi^a_{,m}} - \eta^b_{,k}\frac{\partial L}{\partial\eta^b_{,m}} - \varphi_{1,k}\frac{\partial L}{\partial \varphi_{1,m}} - \varphi_{2,k}\frac{\partial L}{\partial\varphi_{2,m}} + L\delta^m_k\Big). \end{aligned} \quad (3.6)$$

The partial derivatives in (3.6) are found with allowance for relations (2.5):

$$-\xi^a_{,k}\frac{\partial L}{\partial \xi^a_t} = \xi^a_{,k}x^m_{,a}\Big(\rho_1 u_{1m} + \frac{\partial W}{\partial w^m} - \rho_1\varphi_{1,m}\Big) = \rho_1 u_{1k} + \frac{\partial W}{\partial w^k} - \rho_1\varphi_{1,k},$$

$$-\xi^a_{,k}\frac{\partial L}{\partial \xi^a_{,m}} = \xi^a_{,k}x^n_{,a}u_1^m\Big(\rho_1 u_{1n} + \frac{\partial W}{\partial w^n} - \rho_1\varphi_{1,n}\Big) = \rho_1 u_{1k}u_1^m + \frac{\partial W}{\partial w^k}u_1^m - \rho_1\varphi_{1,k}u_1^m.$$

Similarly,

$$-\eta^b_{,k}\,\frac{\partial L}{\partial \eta^b_t} = \rho_2 u_{2k} - \frac{\partial W}{\partial w^k} - \rho_2\varphi_{2,k}, \qquad -\varphi_{1,k}\,\frac{\partial L}{\partial \varphi_{1t}} = \varphi_{1,k}\rho_1,$$

$$-\eta^b_{,k}\,\frac{\partial L}{\partial \eta^b_{,m}} = \rho_2 u_{2k} u_2^m - \frac{\partial W}{\partial w^k}\,u_2^m - \rho_2\varphi_{2,k}u_2^m, \qquad -\varphi_{2,k}\,\frac{\partial L}{\partial \varphi_{2t}} = \varphi_{2,k}\rho_2,$$

$$-\varphi_{1,k}\,\frac{\partial L}{\partial \varphi_{1,m}} = \rho_1\varphi_{1,k}u_1^m, \qquad -\varphi_{2,k}\,\frac{\partial L}{\partial \varphi_{2,m}} = \rho_2\varphi_{2,k}u_2^m.$$

As a result, the law of conservation of momentum (3.6) takes the form

$$\frac{\partial j_k}{\partial t} + \frac{\partial}{\partial x^m}\Big(\rho_1 u_{1k}u_1^m + \rho_2 u_{2k}u_2^m - \frac{\partial W}{\partial w^k}\,w^m + L\delta_k^m\Big) = 0. \tag{3.7}$$

The energy conservation law reflecting the uniformity of space is described by the expression $-\mathcal{E} = \rho_{1t}L_{\rho_1} + \rho_{2t}L_{\rho_2} + \varphi_{1t}L_{\varphi_1} + \varphi_{2t}L_{\varphi_2} + \xi^a_t L_a + \eta^b_t L_b + \gamma_t L_\gamma$, which also has divergent form

$$\mathcal{E} = \frac{\partial}{\partial t}\Big(\xi^a_t\,\frac{\partial L}{\partial \xi^a_t} + \eta^b_t\,\frac{\partial L}{\partial \eta^b_t} + \varphi_{1t}\,\frac{\partial L}{\partial \varphi_{1t}} + \varphi_{2t}\,\frac{\partial L}{\partial \varphi_{2t}} - L\Big) + \frac{\partial}{\partial x^k}\Big(\xi^a_t\,\frac{\partial L}{\partial \xi^a_{,k}} + \eta^b_t\,\frac{\partial L}{\partial \eta^b_{,k}} + \varphi_{1t}\,\frac{\partial L}{\partial \varphi_{1,k}} + \varphi_{2t}\,\frac{\partial L}{\partial \varphi_{2,k}}\Big). \tag{3.8}$$

Calculation of the partial derivatives in (3.8) gives the energy conservation law in obvious form. We have

$$\xi^a_t\,\frac{\partial L}{\partial \xi^a_t} = \rho_1|\mathbf{u}_1|^2 + \frac{\partial W}{\partial w^k}\,u_1^k - \rho_1\varphi_{1,k}u_1^k, \qquad \varphi_{1t}\,\frac{\partial L}{\partial \varphi_{1t}} = -\varphi_{1t}\rho_1,$$

$$\eta^b_t\,\frac{\partial L}{\partial \eta^b_t} = \rho_2|\mathbf{u}_2|^2 - \frac{\partial W}{\partial w^k}\,u_2^k - \rho_2\varphi_{2,k}u_2^k, \qquad \varphi_{2t}\,\frac{\partial L}{\partial \varphi_{2t}} = -\varphi_{2t}\rho_2,$$

$$\xi^a_t\,\frac{\partial L}{\partial \xi^a_{,k}} = \rho_1|\mathbf{u}_1|^2u_1^k + \frac{\partial W}{\partial w^m}\,u_1^m u_1^k - \rho_1\varphi_{1,m}u_1^m u_1^k, \qquad \varphi_{1t}\,\frac{\partial L}{\partial \varphi_{1,k}} = -\rho_1\varphi_{1t}u_1^k,$$

$$\eta^b_t\,\frac{\partial L}{\partial \eta^b_{,k}} = \rho_2|\mathbf{u}_2|^2u_2^k - \frac{\partial W}{\partial w^m}\,u_2^m u_2^k - \rho_2\varphi_{2,m}u_2^m u_2^k, \qquad \varphi_{2t}\,\frac{\partial L}{\partial \varphi_{2,k}} = -\rho_2\varphi_{2t}u_2^k.$$

Hence,

$$\mathcal{E} = \frac{\partial}{\partial t}\Big(U + \frac{\rho_1|\mathbf{u}_1|^2}{2} + \frac{\rho_2|\mathbf{u}_2|^2}{2}\Big) + \frac{\partial}{\partial x^k}\Big(\rho_1 u_1^k\Big(\frac{|\mathbf{u}_1|^2}{2} + \frac{\partial W}{\partial \rho_1}\Big)$$

$$+ \rho_2 u_2^k\Big(\frac{|\mathbf{u}_2|^2}{2} + \frac{\partial W}{\partial \rho_2}\Big) + \frac{\partial W}{\partial w^m}\,(u_1^m u_1^k - u_2^m u_2^k) - \gamma(\alpha_1 u_1^k + \alpha_2 u_2^k)\Big) = 0.$$

Expressing γ in terms of the known thermodynamic functions (3.5), we obtain the energy conservation law for a two-velocity medium with incompressible components:

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x^k}\Big(\rho_1 u_1^k\Big(\frac{|\mathbf{u}_1|^2}{2} + \frac{\partial W}{\partial \rho_1}\Big) + \rho_2 u_2^k\Big(\frac{|\mathbf{u}_2|^2}{2} + \frac{\partial W}{\partial \rho_2}\Big)$$

$$+ \frac{\partial W}{\partial w^m}\,(u_1^m u_1^k - u_2^m u_2^k) + \Big(p + W - \rho_1\,\frac{\partial W}{\partial \rho_1} - \rho_2\,\frac{\partial W}{\partial \rho_2}\Big)(\alpha_1 u_1^k + \alpha_2 u_2^k)\Big) = 0. \tag{3.9}$$

The system of conservation laws for a two-velocity heterogeneous incompressible medium (2.2), (3.7), and (3.9) is written in vector form

$$\frac{\partial \rho_1}{\partial t} + \operatorname{div}(\rho_1\mathbf{u}_1) = 0, \qquad \frac{\partial \rho_2}{\partial t} + \operatorname{div}(\rho_2\mathbf{u}_2) = 0,$$

$$\frac{\partial}{\partial t}(\rho_1\mathbf{u}_1 + \rho_2\mathbf{u}_2) + \operatorname{div}\Big(\rho_1\mathbf{u}_1\otimes\mathbf{u}_1 + \rho_2\mathbf{u}_2\otimes\mathbf{u}_2 - \frac{\partial W}{\partial \mathbf{w}}\otimes\mathbf{w} + Ip\Big) = 0,$$

$$\frac{\partial}{\partial t}\Big(\frac{1}{2}\rho_1|\mathbf{u}_1|^2+\frac{1}{2}\rho_2|\mathbf{u}_2|^2+W-\mathbf{w}\frac{\partial W}{\partial \mathbf{w}}\Big)+\operatorname{div}\Big(\rho_1\mathbf{u}_1\Big(\frac{|\mathbf{u}_1|^2}{2}+\frac{\partial W}{\partial\rho_1}\Big)+\rho_2\mathbf{u}_2\Big(\frac{|\mathbf{u}_2|^2}{2}+\frac{\partial W}{\partial\rho_2}\Big)$$

$$-(\mathbf{u}_2\otimes\mathbf{u}_2-\mathbf{u}_1\otimes\mathbf{u}_1)\Big\langle\frac{\partial W}{\partial\mathbf{w}}\Big\rangle-\Big(p+W-\rho_1\frac{\partial W}{\partial\rho_1}-\rho_2\frac{\partial W}{\partial\rho_2}\Big)(\alpha_1\mathbf{u}_1+\alpha_2\mathbf{u}_2)\Big)=0.$$

System of Equations for Plane Waves. To study the problem of the stability of the medium described, we examine the propagation of plane waves in an one-dimensional approximation. As a simplifying condition we assume that the internal energy is a quadratic function of the relative velocity of the components (condition **A**) $U=\varepsilon(\rho_1,\rho_2)+aw^2/2$ and $W=\varepsilon(\rho_1,\rho_2)-aw^2/2$, where a is a positive constant.

For incompressible media, ρ_1 and ρ_2 are not independent functions: $\rho_1=\hat\rho_1\alpha$ and $\rho_2=\hat\rho_2(1-\alpha)$, where $\alpha\equiv\alpha_1$ and $\alpha_2=1-\alpha$. As a consequence, the thermodynamic system is a two-parameter system:

$$dU=Z\,d\alpha+aw\,dw,\qquad Z=Z(\alpha)=\Big(\frac{\partial\varepsilon}{\partial\rho_1}\Big)\hat\rho_1-\Big(\frac{\partial\varepsilon}{\partial\rho_2}\Big)\hat\rho_2. \tag{3.10}$$

In what follows, we restrict ourselves to the one-dimensional case of system (2.2), (3.7), and (3.9). Within the framework of the adopted assumptions, the laws of conservation of mass (2.2) reduces, by virtue of the geometrical relation (3.1), to the form

$$\alpha_t+(\alpha u_1)_x=0; \tag{3.11}$$

$$(\alpha u_1+(1-\alpha)u_2)_x=0. \tag{3.12}$$

From Eq. (3.12) it follows that value of the expression $\alpha u_1+(1-\alpha)u_2$ does not depend on x. Assuming that the sources of mass are absent at infinity, we set

$$\alpha u_1+(1-\alpha)u_2=0. \tag{3.13}$$

Then, Eq. (3.11) and the energy conservation law form the independent subsystem

$$\alpha_t+(\alpha u_1)_x=0,$$

$$U_t+\Big(\rho_1\frac{u_1^2}{2}\Big)_t+\Big(\rho_2\frac{u_2^2}{2}\Big)_t+\Big(\rho_1u_1\Big(\frac{u_1^2}{2}+\frac{\partial W}{\partial\rho_1}\Big)+\rho_2u_2\Big(\frac{u_2^2}{2}+\frac{\partial W}{\partial\rho_2}\Big)+aw(u_2^2-u_1^2)\Big)_x=0$$

with additional constraint (3.13).

From condition (3.13), it follows that

$$u_1=-(1-\alpha)w,\qquad u_2=\alpha w,\qquad w=u_2-u_1. \tag{3.14}$$

Using (3.14), we arrive at the following system of two equations for the variables α and w:

$$\alpha_t-(\alpha(1-\alpha)w)_x=0; \tag{3.15}$$

$$Z\alpha_t+aww_t+\Big(\hat\rho_1\alpha(1-\alpha)^2\frac{w^2}{2}+\hat\rho_2\alpha^2(1-\alpha)\frac{w^2}{2}\Big)_t-\Big(a(1-2\alpha)w^3\Big)_x$$

$$-\Big(\hat\rho_1\alpha(1-\alpha)^3\frac{w^3}{2}-\hat\rho_2\alpha^3(1-\alpha)\frac{w^3}{2}+\alpha(1-\alpha)\,w\,Z\Big)_x=0 \tag{3.16}$$

[the function Z is introduced in (3.10)].

System (3.15) and (3.16) is equivalent to the equations

$$\alpha_t-\alpha(1-\alpha)w_x-(1-2\alpha)\,w\,\alpha_x=0; \tag{3.17}$$

$$(\hat\rho_1(1-\alpha)+\hat\rho_2\alpha+\tilde a)\,w_t+(\hat\rho_2-\hat\rho_1)w\,\alpha_t+(-\hat\rho_1(1-\alpha)^2+\hat\rho_2\alpha^2$$

$$-3\tilde a(1-2\alpha))w\,w_x+(-Z_\alpha+(\hat\rho_1(1-\alpha)+\hat\rho_2\alpha+2\tilde a)w^2)\,\alpha_x=0, \tag{3.18}$$

where $\tilde a=a/\alpha(1-\alpha)$ and $Z_\alpha=\partial Z/\partial\alpha$. Solving Eqs. (3.17) and (3.18) for the time derivatives, we finally obtain

$$\alpha_t - \alpha(1-\alpha)w_x - (1-2\alpha)w\,\alpha_x = 0; \tag{3.19}$$

$$w_t - \frac{Z_\alpha}{\Delta}\alpha_x - \frac{1}{\Delta}\left(\hat\rho_1(1-\alpha) - \hat\rho_2\alpha + 3\tilde a(1-2\alpha)\right) w\, w_x + \frac{1}{\Delta}\left(\hat\rho_1\alpha + \hat\rho_2(1-\alpha) + 2\tilde a\right) w^2\,\alpha_x = 0, \tag{3.20}$$

where $\Delta = \hat\rho_1(1-\alpha) + \hat\rho_2\alpha + a/\alpha(1-\alpha)$. We note that $\Delta > 0$ for all values $1 \geqslant \alpha \geqslant 0$ and $a > 0$.

Hyperbolicity of the Equations of Motion of Two-Velocity Heterogeneous Media. System (3.19) and (3.20) can be written in matrix form

$$\mathbf{u}_t - A\mathbf{u}_x = 0, \qquad \mathbf{u} = (\alpha, w)^t, \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \tag{3.21}$$

where

$$a_{11} = (1-2\alpha)w, \qquad a_{21} = \frac{1}{\Delta}Z_\alpha - \frac{1}{\Delta}\left(\hat\rho_1\alpha + \hat\rho_2(1-\alpha) + 2\tilde a\right)w^2,$$

$$a_{12} = \alpha(1-\alpha), \qquad a_{22} = \frac{1}{\Delta}\left(\hat\rho_1(1-\alpha) - \hat\rho_2\alpha + 3\tilde a(1-2\alpha)\right)w.$$

The eigenvalues of the matrix A are determined from the equation $\det|A - \lambda I| = 0$:

$$\lambda^2 - (a_{11} + a_{22})\lambda - \left(a_{12}a_{21} - a_{11}a_{22}\right) = 0. \tag{3.22}$$

For roots of Eq. (3.22) to be real, it is necessary that the discriminant

$$D = (a_{11} + a_{22})^2 + 4(a_{12}a_{21} - a_{11}a_{22}) = (a_{11} - a_{22})^2 + 4a_{12}a_{21}$$

be positive. In D we separate terms that depend on the relative velocity:

$$D = 4\frac{\alpha(1-\alpha)}{\Delta}Z_\alpha + 4\frac{w^2}{\Delta^2}\tilde D,$$
$$\tilde D = \alpha(1-\alpha)\Big(\tilde a^2\,\frac{1-6\alpha(1-\alpha)}{\alpha(1-\alpha)} - 3\tilde a\hat\rho_1\alpha - 3\tilde a\hat\rho_2(1-\alpha) - \hat\rho_1\hat\rho_2\Big). \tag{3.23}$$

From (3.23) it follows that in the absence of the relative velocity, system (3.21) is hyperbolic. Indeed, the condition of hyperbolicity reduces to the condition of positiveness of the expression

$$\left(\frac{\partial Z}{\partial\alpha}\right) = \left(\frac{\partial^2 U}{\partial\alpha^2}\right) = \varepsilon_{11}\hat\rho_1^2 - 2\varepsilon_{12}\hat\rho_1\hat\rho_2 + \varepsilon_{22}\hat\rho_2^2 > 0,$$

which corresponds to the condition of thermodynamic stability of the material.

Generally, the condition of hyperbolicity of system (3.21) has the form

$$\left(\frac{\partial Z}{\partial\alpha}\right) > -\frac{w^2}{\Delta}\,\frac{\tilde D}{\alpha(1-\alpha)}.$$

If $\tilde D > 0$, discriminant (3.23) is positive. We consider the expression $\alpha^2(1-\alpha)^2\tilde D$. The condition of positiveness of $\tilde D$ is equivalent to the positiveness of the second-order polynomial in a: $h(a) = pa^2 - qa - r > 0$, where $p = 1 - 6\alpha(1-\alpha)$, $q = 3\alpha^2(1-\alpha)^2(\hat\rho_1\alpha + \hat\rho_2(1-\alpha)) > 0$, and $r = \hat\rho_1\hat\rho_2\alpha^3(1-\alpha)^3 > 0$.

For each fixed $a > 0$, there is apparently a region of volumetric concentrations α (in a small neighborhood of values $\alpha = 0$ and $\alpha = 1$) for which $h(a)$ is positive, and, hence, system (3.21) is hyperbolic. When α belongs to the interval $[1/2 - 1/(2\sqrt3), 1/2 + 1/(2\sqrt3)]$, the function $h(a)$ is negative ($p < 0$) and the model ceases to be hyperbolic for a great velocity difference.

Conclusion. A generalized Hamilton variational principle of the mechanics of two-velocity media is proposed. A Lagrange function is constructed as the difference between the kinetic energy of an element of the medium and the thermodynamic potential, which is the conjugate of internal energy with respect to hydrodynamic variables. This definition of a Lagrangian is general for media whose internal energy depends on thermodynamic variables having the sense of time derivatives.

The generalized Hamilton variational principle was used to formulate the equations of motion of homogeneous and heterogeneous two-velocity continua. Divergent laws of conservation of the total momentum and total energy of the medium are deduced.

It is proved that the convexity of the internal energy ensures the hyperbolicity of the one-dimensional plane-wave flow equations linearized for the state of rest. Thus, the internal energy is a function of both the phase densities and the modulus of the phase-velocity difference.

It is proved that the dependence of the internal energy on the modulus of the relative velocity ensures that the equations of heterogeneous media with incompressible components are hyperbolic for low volumetric concentrations of the phases and any relative velocity of motion of the phases.

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